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INTERFACIAL FRACTURE OF A RADIALLY INHOMOGENEOUS ELASTIC BIMATERIAL

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Abstract—Anti-plane strain crack problems in radially inhomogeneous materials are considered; the model is used as a simple analogy with a non-linear power law material. Both an asymptotic and a rigorous analysis are used to determine the crack tip behaviour. A combined Mellin transform-difference equation method is used to solve the problem of a crack at an interface between a uniform elastic medium and a radially inhomogeneous material. These rigorous results are compared with asymptotic non-linear bimaterial results of Champion and Atkinson [*Proc. R. Soc. Lond.* **A429,** 247–257 (1990)]. Some of the crack tip behaviour associated with the non-linear power law material is shown to be due to the apparent modulus tending to zero or infinity at the crack tip. Problems of cracks at an arbitrary angle to the interface can also be considered using this approach. We briefly consider such cases. To emphasize further similarities between the radially inhomogeneous and the non-linear material we derive some integral invariants.

1. INTRODUCTION

Recently, much attention has been given to the field associated with a crack at an interface between materials with non-linear stress strain laws. It has been shown by Champion and Atkinson (1990) that for power law materials all the energy flow into the crack tip goes into the part of the crack tip lying in the material with the largest hardening exponent. The analysis of Champion and Atkinson is asymptotic. By considering expansions at the crack tip in each region, and matching across the interface ahead of the crack they found the stress singularities are identical in each material. The stress singularity corresponds to that for a crack in a homogeneous material with hardening exponent equal to the maximum hardening exponent of the two materials (the smaller n in the notation used here). Displacements near the crack tip were found to be of different orders in the two materials.

Here we give both a rigorous and asymptotic analysis of an anti-plane strain model problem where the elastic modulus in one medium tends to infinity, or zero, as one approaches the crack tip. This is analogous to the situation in the corresponding non-linear problem where the stress–strain law takes the form $\sigma \sim \varepsilon^n$, $0 < n < \infty$. In this non-linear case a singular strain field will produce an elastic modulus $\mu(\varepsilon)$ [defined by $\sigma = \mu(\varepsilon)\varepsilon$], given as $\mu(\varepsilon) = \varepsilon^{n-1}$, $0 < n < \infty$. For a crack tip in a homogeneous material the energy density $W \sim O(1/r)$, and since $W = \int \sigma \, d\varepsilon \sim \varepsilon^{n+1}$, the strain $\varepsilon \sim O(r^{-1/(n+1)})$ at the crack tip. Thus, if $0 < n < 1$, the analogy is an elastic material with a modulus tending to zero as r tends to the crack tip; if $n > 1$, the analogy is an elastic material with a modulus tending to infinity at the crack tip. When $n = 1$ the stress–strain law corresponds to a linear homogeneous material. Note Champion and Atkinson (1990) write the stress–strain relations in the form $\gamma = \alpha\tau^{n_i}$ (γ is strain and τ stress) so the exponent n_i used there corresponds to $1/n$ used here. The modulus, $\mu(\varepsilon)$, varies with respect to both r, radial distance from the crack tip, and θ, angle subtended by a point relative to the fracture plane. In our inhomogeneous model problem we allow the modulus to vary only with respect to r.

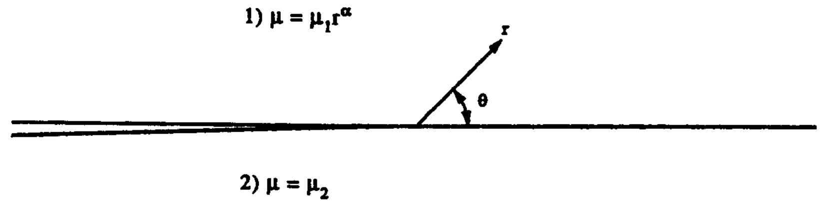

Fig. 1. Bimaterial interface crack: medium 1. Elastic modulus = $\mu_1 r^\alpha$, medium 2. Elastic modulus = μ_2.

Let us briefly consider cracks in a material whose shear modulus varies as r^α in a polar coordinate system centered on the crack tip, the governing equation is $\nabla \cdot r^\alpha \nabla \phi = 0$, where ϕ is the displacement. If the crack faces ($\theta = \pm\pi$, for all r) are stress free then $\phi \sim r^\lambda \cos((\lambda(\lambda+\alpha))^{1/2}(\pi-\theta))$ at the crack tip. The parameter λ is defined by $\lambda = \frac{1}{2}(-\alpha+(\alpha^2+1)^{1/2})$ for $\lambda > 0$, i.e. non-singular and vanishing crack tip displacements. Other eigensolutions can be found for differing values of λ, however these produce solutions for the stresses with no singularity or are unphysically singular as the crack tip is approached. The stress at the crack tip is of order $r^{\alpha+\lambda-1}$, from this we determine that the range of values $0 < \alpha < 3/4$ corresponds to materials whose modulus tends to zero at singular crack tips, and $\alpha < 0$ corresponds to materials whose moduli tend to infinity at singular crack tips.

In the non-linear power law material case the shear modulus is $|\nabla\phi|^{n-1}$. Here we know the strain, $\varepsilon \sim r^{-1/(n+1)}H(\theta)$ and hence the displacement $\phi \sim r^{n/(n+1)}G(\theta)$ at the crack tip. The $G(\theta)$ and $H(\theta)$ are unknown functions of θ. The range $0 < n < 1$ corresponds to the range $0 < \alpha < 3/4$, and $n > 1$ to $\alpha < 0$.

For comparison with the non-linear problem we consider two problems, a homogeneous elastic material (shear modulus μ_2) bonded to a radially inhomogeneous elastic material (shear modulus $\mu_1 r^\alpha$), Fig. 1, and an elastic material bonded to a non-linear power law material obeying the law $\nabla \cdot (|\nabla\phi_1|^{n-1}\nabla\phi_1) = 0$, Fig. 2, where the ϕ_i are displacements in the respective half spaces. The first problem is treated rigorously in this paper using Mellin transform and difference equation techniques. The first problem is also considered using the same asymptotic methods as used in the non-linear problem by Champion and Atkinson (1990). It is interesting to note the similarity between the non-linear and radially inhomogeneous problems; in the radially inhomogeneous case we can treat the problem rigorously and compare it to the other case where the exact non-linear problem is treated asymptotically. Our aim is to highlight the similarity between the problems and crack tip results, showing some of the results associated with the non-linear problem are due, at least in part, to the apparent modulus variation. The asymptotic method used by Champion and Atkinson (1990) in the non-linear problem is applied here to the inhomogeneous problem, the Mellin transform-difference equation method is used to verify that the asymptotic method gives the correct results.

To further compare and contrast the two different problems we briefly consider invariant integrals which can be derived for the radially inhomogeneous material and the non-linear power law material.

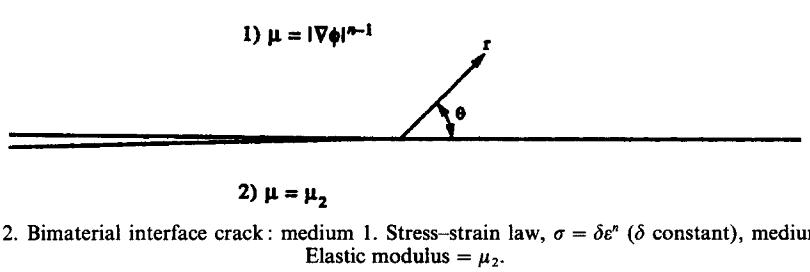

Fig. 2. Bimaterial interface crack: medium 1. Stress–strain law, $\sigma = \delta\varepsilon^n$ (δ constant), medium 2. Elastic modulus = μ_2.

We also consider cracks at an arbitrary angle to the interface; in particular we treat a crack perpendicular to the interface using similar techniques.

2. THE MODEL PROBLEM

Firstly, let us consider $\alpha < 0$ then in Fig. 1 the radially inhomogeneous material behaves like a rigid material (infinite modulus) at the crack tip and the singularity in the elastic material is the usual square root singularity. In the non-linear material the corresponding behaviour is $n > 1$. Note, moreover, if we compare the powers of r in the "effective" modulus, $\alpha = [-(n-1)/(n+1)]$, is the analogy.

Conversely, if we take $0 < \alpha < 3/4$ the inhomogeneous material has a zero shear modulus at the crack tip, this is the somewhat unusual situation of a crack at a free surface, we consequently might expect a stronger singularity at the crack tip than the usual square root case. In the non-linear material the corresponding material behaviour is $0 < n < 1$, where again $\alpha = [-(n-1)/(n+1)]$ is the analogy. We repeat that in this analogy the θ variation of the effective modulus is missing, but it will be of interest to see how the results of the model problem considered here compares with that of Champion and Atkinson (1990).

Moreover, it is worth noting in anticipation for the latter case that the non-linear problem would lead to a weaker singularity in the linear elastic material than the usual inverse square root singularity. This is explained by Champion and Atkinson (1990) in terms of the necessity of matching the singular stresses across the unbroken interface ahead of the crack. They show the stress singularities are the same in each material. The singularity corresponds to that for a crack in a homogeneous material with hardening exponent equal to the maximum of the two materials (recall this means the smaller value of n). Consequently the singularity will be the least singular of the two homogeneous crack tip results.

2.1. *An asymptotic approach*

We use a similar asymptotic method for the radially inhomogeneous interfacial problem to that used in the non-linear problem by Champion and Atkinson (1990). We assume the crack faces on $\theta = \pm\pi$ are stress free, Fig. 1, then in medium 1

$$\phi_1 \sim \sum_{i=1}^{\infty} B_i^{(1)} r^{\lambda_i^{(1)}} \cos\left((\lambda_i^{(1)}(\lambda_i^{(1)}+\alpha))^{1/2}(\pi-\theta)\right), \quad 0 < \theta < \pi \tag{1}$$

and in medium 2

$$\phi_2 \sim \sum_{i=1}^{\infty} B_i^{(2)} r^{\lambda_i^{(2)}} \cos \lambda_i^{(2)}(\pi+\theta), \quad -\pi < \theta < 0. \tag{2}$$

The $B_i^{(1)}$ and $B_i^{(2)}$ are constants. The first of these two equations is the sum of separable solutions of the crack problems in an infinite inhomogeneous body, the second is the sum of separable solutions of the elastic problem in an infinite body. Each solution satisfies the stress free crack face condition. The boundary conditions on $\theta = 0$ are $\phi_1 = \phi_2$ and

$$\frac{\mu_1}{\mu_2} r^{\alpha} \frac{\partial \phi_1}{\partial \theta} = \frac{\partial \phi_2}{\partial \theta}.$$

These conditions are used to fix the values of $\lambda_i^{(j)}$ with $j = 1, 2$.

The asymptotic procedure, for $\alpha < 0$, is that we expect the displacement in medium 2, ϕ_2, to leading order to be

$$\phi_2 \sim B_1^{(2)} r^{1/2} \cos \tfrac{1}{2}(\theta+\pi) \tag{3}$$

and we take the displacement in medium 1, ϕ_1, to be

$$\phi_1 \sim B_1^{(1)} r^{\lambda_1^{(1)}} \cos\left((\lambda_1^{(1)}(\lambda_1^{(1)}+\alpha))^{1/2}(\pi-\theta)\right). \tag{4}$$

Matching the stresses across the boundary we determine both $\lambda_1^{(1)}$ and $B_1^{(1)}$, hence we obtain the leading order behaviour of ϕ_1. However, the displacements do not match across the boundary; we take the next term in the expansion for ϕ_2, from (2), which we match with the leading term in ϕ_1. The next term in the expansion of ϕ_1, (1), is determined by matching the stresses across the boundary, we proceed in this manner to obtain more terms iteratively. The leading two terms in each expansion are given by

$$\phi_2 \sim B_1^{(2)}\left(r^{1/2}\cos\frac{1}{2}(\theta+\pi)\right.$$
$$\left. -\frac{1}{2}\frac{\mu_2}{\mu_1}\frac{r^{1/2-\alpha}}{(\frac{1}{2}(\frac{1}{2}-\alpha))^{1/2}}\frac{\cos((\frac{1}{2}(\frac{1}{2}-\alpha))^{1/2}\pi)}{\sin((\frac{1}{2}(\frac{1}{2}-\alpha))^{1/2}\pi)}\frac{\cos((\frac{1}{2}-\alpha)(\theta+\pi))}{\cos((\frac{1}{2}-\alpha)\pi)}+\cdots\right) \tag{5}$$

$$\phi_1 \sim B_1^{(2)}\left(-\frac{r^{1/2-\alpha}}{2}\frac{\mu_2}{\mu_1}\frac{\cos((\frac{1}{2}(\frac{1}{2}-\alpha))^{1/2}(\pi-\theta))}{(\frac{1}{2}(\frac{1}{2}-\alpha))^{1/2}\sin((\frac{1}{2}(\frac{1}{2}-\alpha))^{1/2}\pi)}\right.$$
$$+\frac{r^{1/2-2\alpha}}{2}\left(\frac{\mu_2}{\mu_1}\right)^2\frac{\sin((\frac{1}{2}-\alpha)\pi)}{\cos((\frac{1}{2}-\alpha)\pi)}\frac{\cos((\frac{1}{2}(\frac{1}{2}-\alpha))^{1/2}\pi)}{\sin((\frac{1}{2}(\frac{1}{2}-\alpha))^{1/2}\pi)}$$
$$\left.\times\frac{2^{1/2}}{(\frac{1}{2}-2\alpha)^{1/2}}\frac{\cos(((\frac{1}{2}-\alpha)(\frac{1}{2}-2\alpha))^{1/2}(\pi-\theta))}{\sin(((\frac{1}{2}-\alpha)(\frac{1}{2}-2\alpha))^{1/2}(\pi-\theta))}+\cdots\right). \tag{6}$$

Other eigensolutions can be deduced in a similar manner by replacing (3) with, say, $\phi_2 \sim B_1^{(1)} r^{3/2}\cos\frac{3}{2}(\theta+\pi)$ and proceeding iteratively. In these cases we would obtain solutions with stress singularities less singular than the leading order term given by (3). Although if $\alpha < -1$ this eigensolution would dominate the second term in (5).

For $0 < \alpha < 3/4$ we expect ϕ_1 to leading order to be

$$\phi_1 \sim B_1^{(1)} r^{\lambda_1^{(1)}} \cos\left((\lambda_1^{(1)}(\lambda^{(1)}+\alpha))^{1/2}(\pi-\theta)\right), \tag{7}$$

where $\lambda_1^{(1)} = \frac{1}{2}(-\alpha+(\alpha^2+1)^{1/2})$, and in medium 2 we take

$$\phi_2 \sim B_1^{(2)} r^{\lambda_1^{(2)}} \cos(\lambda_1^{(2)}(\theta+\pi)). \tag{8}$$

Using the same principles as above, we find

$$\phi_1 \sim B_1^{(1)}\left(r^{\lambda_1^{(1)}}\cos(\tfrac{1}{2}(\pi-\theta))\right.$$
$$\left.-\frac{\mu_1}{\mu_2}r^{\lambda_1^{(1)}+\alpha}\left(\frac{\lambda_1^{(1)}}{\lambda_1^{(1)}+\alpha}\right)^{1/2}\frac{\cos(\lambda_1+\alpha)\pi}{\sin(\lambda_1+\alpha)\pi}\frac{\cos(((\lambda_1^{(1)}+\alpha)(\lambda_1^{(1)}+2\alpha))^{1/2}(\pi-\theta))}{\cos(((\lambda_1^{(1)}+\alpha)(\lambda_1^{(1)}+2\alpha))^{1/2}\pi)}+\cdots\right) \tag{9}$$

and

$$\phi_2 \sim \frac{\mu_1}{\mu_2}B_1^{(1)}\left(\frac{\lambda_1^{(1)}}{\lambda_1^{(1)}+\alpha}\right)^{1/2}\left(-r^{\lambda_1^{(1)}+\alpha}\frac{\cos((\lambda_1^{(1)}+\alpha)(\pi+\theta))}{\sin(\lambda_1^{(1)}+\alpha)\pi}+\frac{\mu_1}{\mu_2}r^{\lambda_1^{(1)}+2\alpha}\right.$$
$$\left.\times\frac{\cos(\lambda_1^{(1)}+\alpha)\pi}{\sin(\lambda_1^{(1)}+\alpha)\pi}\frac{\sin(((\lambda_1^{(1)}+\alpha)(\lambda_1^{(1)}+2\alpha))^{1/2}\pi)}{\cos(((\lambda_1^{(1)}+\alpha)(\lambda_1^{(1)}+2\alpha))^{1/2}\pi)}\frac{\cos((\lambda_1^{(1)}+2\alpha)(\pi+\theta))}{(\lambda_1^{(1)}+2\alpha)\sin((\lambda_1^{(1)}+2\alpha)\pi)}+\cdots\right). \tag{10}$$

This is the iterative solution starting from the assumption that ϕ_1 has the form (7), other similar expansions can be derived by taking $\lambda_1^{(1)} = \frac{1}{2}(-\alpha+(\alpha^2+(2N+1)^2)^{1/2})$ in (7), or by starting with ϕ_2 and assuming $\lambda_1^{(2)} = 1, 2, \ldots$. The leading order behaviour of the stresses for these alternative series of eigensolutions is less singular than the one we have chosen in (7).

These solutions are arbitrary up to a multiplicative constant. The exact solution obtained in the next section determines this constant for a particular loading, verifies that the assumptions made when starting the asymptotic procedure are correct, and that the resulting expansion is also correct. The full eigensolutions are also determined.

The important point is the different behaviour at the crack tips in the different media for the different ranges of α. In the next section we will determine the exact eigensolutions and the solution for a crack loaded with an internal stress. We will then compare the solutions with the asymptotic non-linear results.

2.2. *The exact solution*

For the mode 3 inhomogeneous case we have the model problem shown in Fig. 1. In medium 1 the governing equation is

$$\nabla \cdot (r^\alpha \nabla \phi_1) = 0 \tag{11}$$

and in medium 2

$$\nabla^2 \phi_2 = 0. \tag{12}$$

The ϕ_i are the displacements in the respective media. The boundary conditions (for a welded boundary) ahead of the crack are that on $\theta = 0$

$$\phi_1 = \phi_2, \qquad \chi r^\alpha \frac{\partial \phi_1}{\partial \theta} = \frac{\partial \phi_2}{\partial \theta} \tag{13}$$

and on the crack faces

$$\chi r^\alpha \left(\frac{\partial \phi_1}{\partial \theta} \right)_{\theta=\pi} = \left(\frac{\partial \phi_2}{\partial \theta} \right)_{\theta=-\pi} = r g_1(r)/\mu_2 = g(r), \quad \text{for} \quad 0 < r < \infty, \tag{14}$$

where for simplicity an internal stress $g_1(r)$ is taken on the crack faces and $\chi = \mu_1/\mu_2$. The asymptotic problem we considered in the previous section is an eigensolution which corresponds to $g_1(r) = 0$. We define the Mellin transform of $\phi(r, \theta)$ as

$$\Phi(s, \theta) = \int_0^\infty r^{s-1} \phi(r, \theta)\, \mathrm{d}r \tag{15}$$

and the inverse transform as

$$\phi(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi(s, \theta) r^{-s}\, \mathrm{d}s. \tag{16}$$

The constant c is determined from subsidiary conditions based on physical considerations for the particular problem considered. The displacements are required to tend to zero at the crack tip and, if we take a prescribed internal stress loading for the crack, the displacements tend to zero at infinity. Hence c will lie in a strip $0-\delta < \Re(s) < 0+\delta$, where δ is real, positive and its precise value is determined from the positions of the singularities of the transform.

Multiplying the governing equations (11 and 12) by r^{s-1} and integrating with respect to r from 0 to ∞ and solving the resulting Mellin transformed equations gives

$$\Phi_2(s,\theta) = A_2(s)\cos s\theta + B_2(s)\sin s\theta \tag{17}$$

$$\Phi_1(s+\alpha,\theta) = A_1(s+\alpha)\cos\gamma\theta + B_1(s+\alpha)\sin\gamma\theta, \tag{18}$$

where $\gamma^2 = s(s+\alpha)$ and the functions $A_i(s)$, $B_i(s)$ are independent of θ. Let us define $\bar{g}(s)$ as the Mellin transform of $g(r)$; using the boundary conditions we find $\Phi_2(s,\theta)$ and $\Phi_1(s,\theta)$ as

$$\Phi_2(s,\theta) = \frac{\bar{g}(s)\sin s\theta}{s\cos s\pi} + A_2(s)\frac{\cos s(\theta+\pi)}{\cos s\pi} \tag{19}$$

$$\Phi_1(s+\alpha,\theta) = \frac{\bar{g}(s)\sin\gamma\theta}{\chi\gamma\cos\gamma\pi} + A_2(s+\alpha)\frac{\cos\gamma(\theta-\pi)}{\cos\gamma\pi}. \tag{20}$$

In the case of two homogeneous materials welded together, the function $A_2(s) = 0$; the remaining transforms indicate the displacement ahead of the crack is identically zero. For the problem considered here, i.e. (13) and (14), the zero displacement on the welded boundary is maintained by the first terms in eqns (19) and (20). These are the terms we would get by solving each of the mode 3 problems in each half space in the absence of the other half space. The terms involving $A_2(s)$ couple the behaviour of the two half spaces together.

From the boundary conditions the following difference equation for $A_2(s)$ can be deduced

$$s\sin s\pi\cos\gamma\pi A_2(s) + \chi A_2(s+\alpha)\gamma\sin\gamma\pi\cos s\pi = \bar{g}(s)\,(\cos\gamma\pi - \cos s\pi). \tag{21}$$

We get a formal solution for the difference equation (21) by initially solving a homogeneous equation

$$A_2^*(s+\alpha) = \frac{-s\sin s\pi\cos\gamma\pi}{\chi\gamma\sin\gamma\pi\cos s\pi}A_2^*(s). \tag{22}$$

The full difference equation (21) is then solved by using this homogeneous solution together with a particular solution of the different equation which is determined later.

We set

$$A_2^*(s) = \chi^{-s/\alpha}X(s)/\sin\frac{\pi s}{\alpha},$$

a subsidiary equation for $X(s)$ is found as

$$X(s+\alpha) = K(s)X(s), \tag{23}$$

where the function $K(s)$ is given by

$$K(s) = \frac{s\sin s\pi\cos\gamma\pi}{\gamma\sin\gamma\pi\cos s\pi}. \tag{24}$$

$K(s)$ is single valued and has the appropriate convergence properties we require in the strip $-\alpha \leqslant \Re(s) < \alpha$, see the Appendix. Hence, we can, via an application of the Plemelj formulae (Bantsuri, 1973) find a solution valid in the strip $-\alpha \leqslant \Re(s) < \alpha$. This solution is given by

$$X(s) = Y(s), \quad 0 \leqslant \Re(s) < \alpha \tag{25}$$

$$X(s) = Y(s)/K(s), \quad -\alpha \leqslant \Re(s) < 0. \tag{26}$$

The function $Y(s)$ is given by

$$\log Y(s) = \frac{1}{2i\alpha}\int_{-i\infty}^{i\infty} \cot\left(\frac{\pi(t-s)}{\alpha}\right)\log K(t)\,\mathrm{d}t \tag{27}$$

which is a Cauchy principal value integral. Note this solution is also valid for α negative. The method required is described in some detail in the Appendix. The solution to the homogeneous equation is arbitrary up to a multiplicative function of period α. Outside the strip the full solutions can be determined by using the original difference equation to deduce that for $0 < \alpha \leqslant 3/4$ in $-(n+1)\alpha \leqslant \Re(s) < -n\alpha$

$$A_2^*(s) = \frac{Y(s)}{\chi^{s/\alpha}\sin(\pi s/\alpha)\prod_{k=0}^{n} K(s+k\alpha)} \tag{28}$$

and in $n\alpha \leqslant \Re(s) < (n+1)\alpha$

$$A_2^*(s) = \frac{Y(s)}{\chi^{s/\alpha}\sin(\pi s/\alpha)}\prod_{k=1}^{n} K(s-k\alpha). \tag{29}$$

If $n = 0$ in (29) the product term is one. The above formulae hold for α negative with minor changes. The position of the poles and zeros of $A^*(s)$ can be determined using (28) and (29).

The eigensolution, which we examined using the asymptotic method in the previous section, is given rigorously by

$$\Phi_2(s,\theta) = A_2^*(s)\frac{\cos s(\theta+\pi)}{\cos s\pi} \tag{30}$$

$$\Phi_1(s+\alpha,\theta) = A_2^*(s+\alpha)\frac{\cos\gamma(\theta-\pi)}{\cos\gamma\pi}. \tag{31}$$

The solution for $A_2^*(s)$ can be used to verify the expansions determined in (5), (6), (9) and (10) are correct; the inversion contour in (16) is taken so $\max(-\frac{1}{2}, -\frac{1}{2}(\alpha+(\alpha^2+1)^{1/2})) < \Re(c) < 0$. This verifies that the asymptotic procedure used by Champion and Atkinson (1990) gives the correct solution when applied to the radially inhomogeneous material. As an aside it also shows an alternative method of solving a difference equation could be to proceed iteratively and identify the position and order of the poles and zeros, but this is far less straightforward than the Hilbert approach.

The inhomogeneous difference equation is solved as in Craster and Atkinson (1994) by determining a particular solution, see also Atkinson (1977). We let $A_2(s) = A_2^*(s)A_2^{(0)}(s)$, where the particular solution is given by the following difference equation

$$A_2^{(0)}(s+\alpha) - A_2^{(0)}(s) = \frac{\bar{g}(s)(\cos\gamma\pi - \cos s\pi)}{\gamma\sin\gamma\pi\cos s\pi A_2^*(s+\alpha)} = f(s). \tag{32}$$

The inhomogeneous difference equation (32) is solved by reformulating the difference equation as a periodic Hilbert problem. The solution follows that outlined in the Appendix

$$A_2^{(0)}(s) = \frac{1}{2i\alpha}\int_{-i\infty}^{i\infty} f(t)\cot\frac{\pi(t-s)}{\alpha}\,\mathrm{d}t, \quad 0 \leqslant \Re(s) < \alpha \tag{33}$$

$$A_2^{(0)}(s) = \frac{1}{2i\alpha}\int_{-i\infty}^{i\infty} f(t)\cot\frac{\pi(t-s)}{\alpha}\,\mathrm{d}t - f(s), \quad -\alpha \leqslant \Re(s) < 0. \tag{34}$$

This solution is only valid in the strip $-\alpha \leqslant \Re(s) < \alpha$, however, this can be extended to the whole domain by using the original difference equation (32). The full formal solution of the difference equation (21) is given by

$$A_2(s) = A_2^*(s)(A_2^{(0)}(s) - A_2^{(0)}(0)). \tag{35}$$

The arbitrary function of period one is fixed by using $A_2^{(0)}(0)$ to cancel a simple pole at zero in $A_2^*(s)$. This simple pole is not allowed since from physical constraints on the displacements (e.g. vanishing displacements at infinity) we expect $A_2(s)$ to be analytic in a strip, $\max(-\frac{1}{2}, -\frac{1}{2}(\alpha+(\alpha^2+1)^{1/2})) < \Re(s) < 0+\delta$ $(\delta > 0)$. This analyticity requirement ensures that the solution decays appropriately at infinity. The solution chosen above can be shown to be unique using the Riemann–Lebesque theorem, as only this solution satisfies the subsidiary physical conditions.

Let us for definiteness take the internal loading to be $\sigma\delta(r-l)$; the solution for the uncoupled parts of the displacements in each of the half spaces can be deduced to be

$$\phi_1(r,\theta) = \frac{2\sigma}{\mu_1 \pi l^{\alpha}} \sum_{n=0}^{\infty} \frac{(-1)^n \sin\left(\frac{2n+1}{2}\theta\right)}{(\alpha^2+(2n+1)^2)^{1/2}} \left(\frac{r}{l}\right)^{-(\alpha/2)\pm[(\alpha^2+(2n+1)^2)^{1/2}/2]} \tag{36}$$

with the plus (minus) sign for $r < l$ $(r > l)$, and

$$\phi_2(r,\theta) = \frac{\sigma}{2\pi\mu_2}\log\left[\frac{1+2\left(\frac{r}{l}\right)^{1/2}\sin\frac{\theta}{2}+\frac{r}{l}}{1-2\left(\frac{r}{l}\right)^{1/2}\sin\frac{\theta}{2}+\frac{r}{l}}\right]. \tag{37}$$

These formulae come from evaluating the residue contributions from the poles in the transforms and are valid for any α, we note from these that in material 1, for $0 < \alpha < 3/4$ the exponents in r for the displacements are smaller at the crack tip than those in the elastic case, locally the moduli tend to zero and we have the situation of a crack at a "free" surface. Both of these solutions have the displacement on the fracture plane ahead of the crack identically zero.

The coupling terms contribute terms which to leading order for small r are; for $\alpha < 0$

$$\phi_2(r,\theta) \sim -\frac{r^{1/2}}{\pi}\chi^{(1/2\alpha)}Y\left(-\frac{1}{2}\right)\cos\left(\frac{1}{2}(\theta+\pi)\right)\frac{1}{2i\alpha}\int_{-i\infty}^{i\infty}\frac{f(\tau)\,\mathrm{d}\tau}{\sin((\pi/\alpha)(\tau+\frac{1}{2}))\sin(\pi\tau/\alpha)} \tag{38}$$

and that for $0 < \alpha < 3/4$ on $\theta = 0$

$$\phi_2 \sim -\frac{1}{2}\frac{Y(s')r^{-s'}\cos s'\pi}{(\alpha^2+1)^{1/2}\chi^{s'/\alpha}s'\sin s'\pi}\frac{1}{2i\alpha}\int_{c'-i\infty}^{c'+i\infty}\frac{f(\tau)\,\mathrm{d}\tau}{\sin((\pi/\alpha)(\tau-s'))\sin(\pi\tau/\alpha)} \tag{39}$$

with $s' = -\frac{1}{2}(\alpha+(\alpha^2+1)^{1/2})$ with $\Re(s')-\delta < \Re(c') < \Re(s')$, δ is real and positive. This identifies the leading order behaviour of the displacements at the crack tips explicitly. Higher order terms and terms for $\phi_1(r,\theta)$ follow in a similar manner.

At this point let us consider what happens in the non-linear material case considered by Champion and Atkinson (1990). In their asymptotic approach, the displacement ahead of the crack tip is taken to be zero (to leading order) in the material which has the most singular behaviour. This choice is made on physical grounds. As the stresses have the same singular behaviour at the crack tip in each of the half spaces, then if we did not take the displacement chosen above to be zero, the energy density in the other material would be unphysically singular. It would also not be possible to match the displacements across the boundary and still have non-singular displacements at the crack tip. The asymptotic displacement field is then determined by matching up separable solutions of the governing equations. The result in the half space with the most singular stress–strain law is that the strains are more singular as the crack tip is approached and the energy density is concentrated in that layer.

In the radially inhomogeneous material considered here, each half space has a part of the displacement field which can be identified with each half space independent of the other. The displacements are coupled across the boundary by the terms involving $A_2(s)$. Using the asymptotic and exact solutions derived above we find the radially inhomogeneous material has similar crack tip behaviour to the non-linear material. In particular the displacements have similar behaviour for $\alpha < 0$, $0 < \alpha < 3/4$ to the non-linear material in $n < 1$, $1 < n < \infty$, moreover the stress singularities in each half space are identical and the same as that for a crack in a single material with the more singular stress–strain law. We also observe that the half space with the more singular stress–strain law has more singular strains and the energy density is concentrated in that layer. Hence, we find that some of the behaviour associated with the non-linear material is due, at least in part, to the apparent modulus variation and not necessarily due to the non-linearity of stress–strain law. Having shown the similarity between the crack tip behaviour in this interfacial problem we now briefly consider some more general problems. We have verified explicitly that the variable separation method adopted by Champion and Atkinson is correct for the linear material.

2.3. *Cracks at an arbitrary angle*

For a crack at any angle to an interface between two homogeneous, linear isotropic half spaces opening under anti-plane strain, we recall the stress singularity is a function of the angle of inclination of crack to the interface and of the shear moduli of the elastic materials. The stress is of order $r^{-(s+1)}$ at the crack tip, where s is given by the smallest negative zero (between 0 and -1) of the expression

$$\sin s\left(\beta-\frac{\pi}{2}\right)\sin s\left(\beta+\frac{\pi}{2}\right)+\chi\cos s\left(\beta-\frac{\pi}{2}\right)\cos s\left(\beta+\frac{\pi}{2}\right).$$

It is of course possible, in theory, to extend the analysis of the previous sections to the case of cracks at any angle to the interface. Let us take a crack lying in an elastic material with modulus μ_2 at an angle β to the vertical, i.e. Fig. 3. Mellin transforming the governing equations we deduce that they become

$$\Phi_1(s+\alpha) = A_1(s+\alpha)\cos\gamma\left(\theta+\beta-\frac{\pi}{2}\right)+B_1(s+\alpha)\sin\gamma\left(\theta+\beta-\frac{\pi}{2}\right),\quad 0<\theta<\pi \tag{40}$$

$$\Phi_2(s) = A_2(s)\cos s(\pi-\theta)+B_2(s)\sin s(\pi-\theta),\quad -\pi<\theta<-\frac{\pi}{2}-\beta \tag{41}$$

and

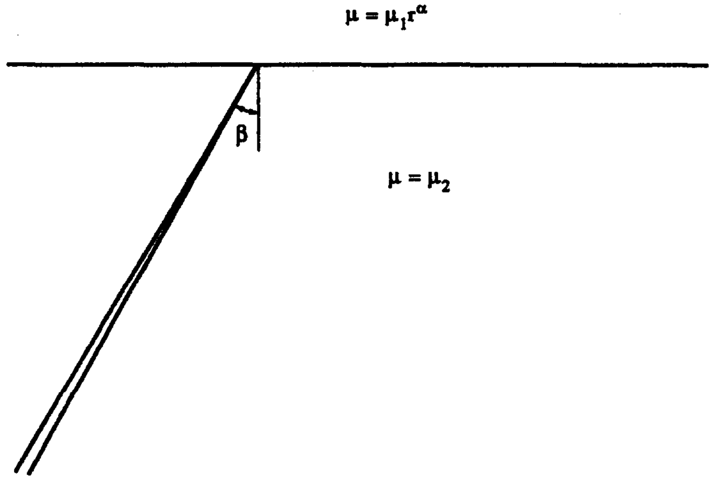

Fig. 3. A crack at an angle to the interface.

$$\Phi_3(s) = A_3(s)\cos s\theta + B_3(s)\sin s\theta, \quad -\frac{\pi}{2}-\beta < \theta < 0. \tag{42}$$

Solving the above for continuity of stress and displacement across the interface and with the crack subjected to an internal stress $g_1(r)$, as before, gives us the following system of difference equations

$$\begin{bmatrix} \sin s\left(\frac{\pi}{2}+\beta\right)\cos\gamma'\left(\beta-\frac{\pi}{2}\right) & \sin\gamma'\left(\beta-\frac{\pi}{2}\right)\sin s\left(\beta+\frac{\pi}{2}\right) \\ \sin s\left(\beta-\frac{\pi}{2}\right)\cos\gamma'\left(\beta+\frac{\pi}{2}\right) & \sin s\left(\beta-\frac{\pi}{2}\right)\sin\gamma'\left(\beta+\frac{\pi}{2}\right) \end{bmatrix} \begin{pmatrix} A_1(s) \\ B_1(s) \end{pmatrix}$$

$$+\frac{\mu_1\gamma}{s\mu_2}\begin{bmatrix} -\cos s\left(\frac{\pi}{2}+\beta\right)\sin\gamma\left(\beta-\frac{\pi}{2}\right) & \cos\gamma\left(\beta-\frac{\pi}{2}\right)\cos s\left(\frac{\pi}{2}+\beta\right) \\ -\sin\gamma\left(\beta+\frac{\pi}{2}\right)\cos s\left(\frac{\pi}{2}-\beta\right) & \cos s\left(\beta-\frac{\pi}{2}\right)\cos\gamma\left(\beta+\frac{\pi}{2}\right) \end{bmatrix} \begin{pmatrix} A_1(s+\alpha) \\ B_1(s+\alpha) \end{pmatrix}$$

$$= \frac{\bar{g}(s)}{s}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{43}$$

The functions $A_2(s)$, $B_2(s)$, $A_3(s)$, $B_3(s)$ can be related in a straightforward manner to $A_1(s)$, $B_1(s)$. The notation $\gamma' = (s(s-\alpha))^{1/2}$. This system of difference equations appears to be unsolvable using the Hilbert transform technique. The iterative method does work, and this may show a constructive method of solving such problems. Only in two cases is there any simplification; in the limit $\beta = \pi/2$ we recover the difference equation considered in Section 2.2. The only other degenerate case is $\beta = 0$, i.e. the crack is perpendicular to the interface; in this case the system uncouples and we obtain the following two difference equations

$$s\sin\frac{s\pi}{2}\cos\frac{\gamma'\pi}{2}A_1(s) + \chi\gamma\cos\frac{s\pi}{2}\cos\frac{\gamma\pi}{2}A_1(s+\alpha) = 0 \tag{44}$$

$$-s\sin\frac{\gamma'\pi}{2}\sin\frac{s\pi}{2}B_1(s) + \chi\gamma\cos\frac{\gamma\pi}{2}\cos\frac{s\pi}{2}B_1(s+\alpha) = \bar{g}(s). \tag{45}$$

It is not clear that these two difference equations can be solved using the Hilbert technique as the kernel functions are not single valued. However, we can pursue the iterative approach as in Section 2.1. We consider a crack in an elastic material, the tip of which is touching a wedge of radially inhomogeneous material of angle $2(\pi-\beta)$ at the vertex (Fig. 4). The iterative solution to this can be deduced, for $\alpha < 0$, as

$$\phi_1 \sim B_2 r^{(\pi/2\beta)-\alpha} \frac{\sin\left(\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}\theta\right)\frac{\pi}{2\beta}}{\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}\cos\left(\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}(\pi-\beta)\right)}+\cdots \tag{46}$$

$$\phi_2 \sim B_2 \left[r^{\pi/2\beta}\cos\frac{\pi}{2\beta}(\pi-\theta) + \frac{r^{(\pi/2\beta)-\alpha}\sin\left(\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}(\pi-\beta)\right)\frac{\pi}{2\beta}}{\cos\left(\frac{\pi}{2\beta}-\alpha\right)\beta\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}\cos\left(\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}(\pi-\beta)\right)}+\cdots\right], \tag{47}$$

in the limit as $\beta \to \pi/2$ there is no stress singularity. We can compare this to the asymptotic

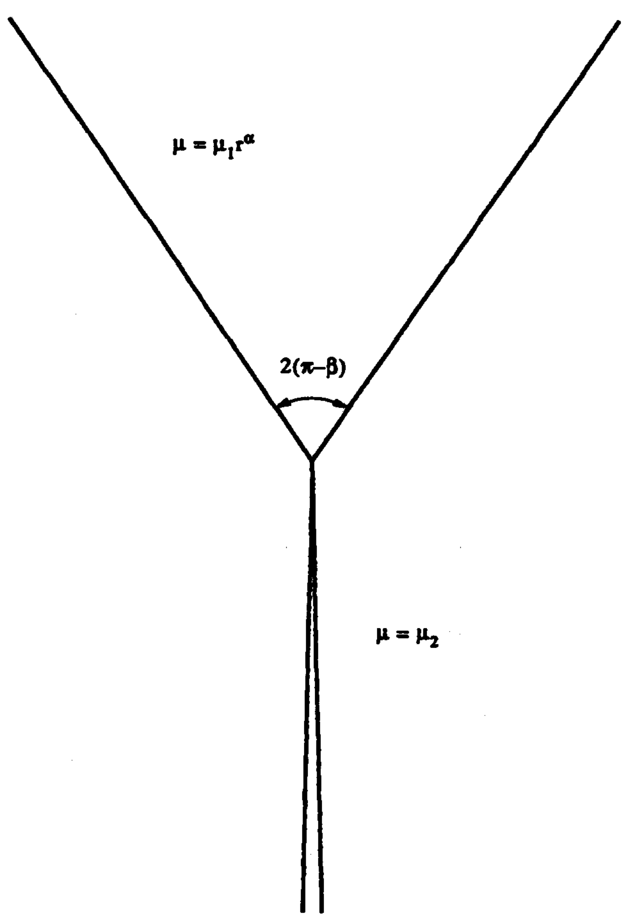

Fig. 4. Crack touching a radially inhomogeneous wedge.

non-linear power law results deduced by Atkinson and Champion (1994), a full comparison is made in Section 4.

3. INVARIANT INTEGRALS

One theme of this paper has been a simple model of the near crack tip behaviour at an interface between two materials whose constitutive behaviour satisfy a non-linear power law stress–strain relation. This non-linear behaviour at the crack tip is modelled by means of a linear elastic model with radially varying elastic modulus of power law type. Qualitative agreement between the singular near crack tip stress fields of the two models can be demonstrated and as shown in the previous section this enables a complete analysis to be made. In the full non-linear problem only an asymptotic analysis is available as in Champion and Atkinson (1990). To further illustrate the similarity between the two formulations, and notice some dissimilarities we derive here some invariants for the two sets of equations and consider their application to the problem at hand.

Firstly, the reciprocal theorem

$$\int_S (\sigma_{3j} u_3^* - \sigma_{3j}^* u_3) n_j \, \mathrm{d}S = 0, \tag{48}$$

where S is a closed surface and the vector n_j is the outward pointing normal, we assume there are no body forces enclosed within S. The reciprocal theorem still holds, of course, for the spatially varying inhomogeneous elastic problem. The starred and unstarred fields are two independent solutions of the governing equations enclosed by a surface S.

However, for the non-linear material considered in Champion and Atkinson (1990) where

$$\sigma_{3j} = \mu_0 (u_{3,k} u_{3,k})^{(n-1)/2} u_{3,j}, \quad k, j = 1, 2 \tag{49}$$

with the usual summation convention and x_j representing the Cartesian coordinates, no such reciprocal theorem can be deduced.

For the non-linear constitutive law (49) well-known invariants exist, for such a material the energy density is

$$W = \frac{\mu_0}{n+1} (u_{3,k} u_{3,k})^{(n+1)/2}. \tag{50}$$

Thus, following Eshelby (1951) we can define the energy momentum tensor P_{lj} as

$$P_{lj} = W\delta_{lj} - \sigma_{3j} u_{3,l}, \quad j, l = 1, 2 \tag{51}$$

it is clear from a direct calculation that $P_{lj,j} = 0$ and hence the integrals

$$F_l = \int_S P_{lj} n_j \, \mathrm{d}S \tag{52}$$

are invariant when S is a surface enclosing no singularities. Other invariants can be deduced, one which is due to the invariance of the equations in radial changes of scale is

$$M = \int_S \left(x_l P_{lj} + \left(\frac{n-1}{n+1} \right) \sigma_{3j} u_3 \right) n_j \, \mathrm{d}S. \tag{53}$$

Because of the spatial inhomogeneity of the simple linear model which has the form

$$\sigma_{3j} = \mu_0 r^\alpha u_{3,j}, \quad \text{where} \quad r^2 = x_k x_k, \quad k = 1,2, \tag{54}$$

it is now shown that no invariant similar to F_l above exists. Nonetheless, as the governing equation

$$\sigma_{3j,j} = (r^\alpha u_{3,j})_{,j} = 0 \tag{55}$$

is invariant due to radial changes in scale, we can construct an integral of the type usually designated by M. For this medium the energy density is given by

$$W = \frac{\mu_0}{2} r^\alpha u_{3,k} u_{3,k}. \tag{56}$$

We define P_{lj} as in (51), then it is clear that

$$P_{lj,j} = \frac{\alpha}{2}\mu_0 x_l r^{\alpha-2} u_{3,k} u_{3,k}. \tag{57}$$

Therefore if $\alpha \neq 0$ no invariant similar to F_l exists. We now define M as

$$M = \int_S \left(x_l P_{lj} - \frac{\alpha}{2} u_3 \sigma_{3j}\right) n_j \, \mathrm{d}S \tag{58}$$

which is invariant for any surface enclosing no singularities.

4. CONCLUSION

We have shown how a combined Mellin transform-difference equation method can be used to solve problems of cracks at an interface between a constant modulus elastic material and a material with a radially varying elastic modulus. These results are compared with the analysis of the non-linear case of an elastic medium bonded to a medium with a power law stress–strain curve. The main results are as follows.

4.1. *Bimaterial interface cracks*

(A) The non-linear cases: medium 1 has a stress–strain law where $\sigma = \delta\varepsilon^n$ (δ constant), and medium 2 is linear elastic and has an elastic modulus of unity.

- If $n > 1$ the leading term in the stress of either medium on the welded boundary has the form $\sigma_{yz}^{(i)} \sim A^{(i)}(\theta) r^{-1/2} + \cdots$, where the $A^{(i)}(\theta)$ are functions of θ appropriate to each medium. These become equal on $\theta = 0$ to a constant value depending on the precise loading conditions. The displacements are given to leading order as

$$\phi_2 \sim a^{(2)}(\theta) r^{1/2}, \qquad \phi_1 \sim a^{(1)}(\theta) r^{1-(1/2n)}. \tag{59}$$

The functions $A^{(i)}(\theta)$, $a^{(i)}(\theta)$ and some higher order terms are given in Champion and Atkinson (1990).

- If $0 < n < 1$ the leading term in the stress of either medium on the welded boundary has the form $\sigma_{yz}^{(i)} \sim A^{(i)}(\theta) r^{-[n/(n+1)]} + \cdots$. The leading order behaviour of the displacements can be determined as

$$\phi_2 \sim a^{(2)}(\theta) r^{1/(n+1)}, \qquad \phi_1 \sim a^{(1)}(\theta) r^{n/(n+1)}, \tag{60}$$

where once again the functions $A^{(i)}(\theta)$, $a^{(i)}(\theta)$ can be determined.

(B) The inhomogeneous model problem considered: medium 1 has an elastic modulus of $\mu_1 r^\alpha$, and medium 2 has an elastic modulus of μ_2.

- If $\alpha < 0$ then to leading order the stress, $\sigma_{yz} \sim A r^{-1/2}$ and the displacements are

$$\phi_2 \sim B^{(2)} r^{1/2} \cos \tfrac{1}{2}(\theta+\pi), \qquad \phi_1 \sim B^{(1)}(\theta) r^{1/2-\alpha}, \tag{61}$$

the function $B^{(1)}(\theta)$ can be determined explicitly.

- If $0 < \alpha < 3/4$ then the stress, $\sigma_{yz} \sim Ar^{\lambda_1^{(1)}+\alpha-1}$ and the displacements are

$$\phi_2 \sim B^{(2)}(\theta) r^{\lambda_1^{(1)}+\alpha}, \qquad \phi_1 \sim B^{(1)} r^{\lambda_1^{(1)}} \cos \tfrac{1}{2}(\pi-\theta). \tag{62}$$

The variable $\lambda^{(1)}$ is given by $\lambda^{(1)} = \frac{1}{2}(-\alpha+(\alpha^2+1)^{1/2})$.

The two models have very similar crack tip behaviour. In both cases the stress singularity at the crack tip is the same as that of a crack in a single material with the more singular stress–strain law. The displacements in each half space have similar behaviour in the analogous cases, although we note that the analogy $\alpha = [-(n-1)/(n+1)]$ is not exact. In this, as in our simple inhomogeneous model, we have ignored any θ variation in the shear modulus. Nonetheless, the main characteristics exhibited by the non-linear problem are also shown by the inhomogeneous model problem thereby showing that some of the phenomena associated with the power law model are due to the apparent modulus variation. In the case when $0 < \alpha < 3/4$, the analogy appears to hold quite strongly, if we take $\alpha = [-(n-1)/(n+1)]$, as a simple comparison of the effective moduli would suggest, then the relative difference between the powers of r in the different half spaces for (60) and (62) are the same. If we further take $\lambda^{(1)} = \frac{1}{2}(1-\alpha)$, the powers of r in (60) and (62) would correspond exactly.

We also verify that the asymptotic method used by Champion and Atkinson (1990) in their analysis of the nonlinear interfacial crack can be used for the radially inhomogeneous problems and that it gives the correct answers when compared with an exact analysis. The exact analysis of the problem identifies the coefficient of the leading order stresses and displacements at the crack tip and provides an example of a problem which is solved using Mellin transform and difference equation techniques.

4.2. *Cracks against interfaces*

To study this case we considered in Section 2.3 the situation shown in Fig. 4. A crack lying in a homogeneous elastic material meets at its vertex a wedge of radially inhomogeneous material, $\mu(r) = r^\alpha$ ($\alpha < 0$) of included angle $2(\pi-\beta)$, ($\beta < \pi/2$). Let us compare the main results of Section 2.3 with corresponding results for the non-linear ($\sigma \sim \varepsilon^n$) case, these have been obtained by Atkinson and Champion (1994). There the inhomogeneous medium above is replaced by $\sigma \sim \varepsilon^n$, $n > 1$ corresponding to $\mu(r) \sim r^\alpha$, $\alpha < 0$. The main results are

(A) The non-linear case: medium 1 has a stress–strain law of $\sigma = \beta\varepsilon^n$ (β constant) and medium 2 is linear elastic. The leading terms in the stress match on the wedge boundary, but the corresponding leading terms in the displacement have different powers of r (measured from the crack tip). It is then deduced that the leading term in the displacement in the elastic medium must be identically zero on the wedge boundary. Thus, in the elastic material the displacements in the neighbourhood of the crack tip are

$$\phi_2 \sim B_2 r^{(\pi/2\beta)} \cos \frac{\pi}{2\beta}(\pi-\theta), \quad \pi-\beta < \theta < \pi \tag{63}$$

and in the non-linear material

$$\phi_1 \sim K_3 r^{1+[(\pi-2\beta)/2\beta n]} F_3(\theta), \quad 0 < \theta < \beta, \tag{64}$$

where $F_3(0) = 0$ and the function $F_3(\theta)$ is given in an implicit form in Atkinson and Champion (1994). The coefficient K_3 is determined in terms of B_2 from matching the traction stresses on the wedge boundary. Higher terms in this expansion are given in Atkinson and Champion (1994). It should be noted that we assumed $\beta > \pi/2$ here. If $\beta = \pi/2$ the exponent of the power of r in both ϕ_1, ϕ_2 is unity, so it looks like no singular stresses are obtained. However a more careful investigation may be required in this limit. We have given results for $0 < \theta < \pi$ only; note that the displacement field is antisymmetric about $\theta = 0$.

(B) The inhomogeneous model problem considered: medium 1 has an elastic modulus of r^{α} and medium 2 has a constant elastic modulus. We take $\alpha < 0$, then using the iterative approach we obtain in the elastic material

$$\phi_2 \sim B_2 r^{\pi/2\beta} \cos\frac{\pi}{2\beta}(\pi-\theta)+\cdots \tag{65}$$

and in the radially inhomogeneous material

$$\phi_1 \sim B_2 r^{(\pi/2\beta)-\alpha} \frac{\sin\left(\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}\theta\right)\frac{\pi}{2\beta}}{\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}\cos\left(\left(\left(\frac{\pi}{2\beta}-\alpha\right)\frac{\pi}{2\beta}\right)^{1/2}(\pi-\beta)\right)}+\cdots. \tag{66}$$

As the asymptotic variable separable method used by Champion and Atkinson (1990) leads to the correct solution in the inhomogeneous interface problem, we conclude that the approach is well founded. Earlier analysis of the plane strain non-linear interface problems by Shih and Asaro (1988) had concluded that variable separable solutions of these non-linear problems probably did not exist. As the asymptotic method of Champion and Atkinson (1990) which led to variable separable solutions is not rigorous, it is reassuring that for the closely related inhomogeneous interface problem the method works exactly. We should note that in this paper we have considered only anti-plane problems and that Champion and Atkinson (1991) have extended their asymptotic analysis to the plane strain interface problems. Recently Sharma and Aravas (1993) have extended the asymptotic work of Champion and Atkinson (1991) to determine higher order terms, furthermore they verify (numerically) that the asymptotic results are correct.

It is likely that in plane strain a similar analogy with a radially inhomogeneous elastic material exists, although the analysis would become much more complicated.

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APPENDIX: THE SOLUTION OF FIRST-ORDER LINEAR DIFFERENCE EQUATIONS

In this appendix we briefly present some of the technical results required for the solution of the first order difference equations used in the text. Take a homogeneous difference equation

$$A(s+\alpha) = F(s)A(s). \tag{A1}$$

If $F(s)$ is just a simple trigonometric function, one can deduce the solution merely by inspection and using the appropriate properties of Alexeiewsky's function (or the double gamma function if necessary). The solution thus deduced will be arbitrary up to a multiplicative constant of period one which is fixed from the physical constraints of the problem. However, when the function $F(s)$ is more complicated one can proceed as in Barnes (1904) by decomposing $F(s)$ into its Weierstrass products. This can then be written in terms of an infinite product of gamma functions with appropriate exponential convergence factors. This is somewhat unwieldy and tedious, the exponential convergence factors required are awkward to derive. Stoker (1957) solved q-difference equations, these are analogous to linear difference equations, using a Cauchy integral technique. To apply this technique to linear difference equations requires a simple change of variable. However, the method is less direct and requires some complicated arguments, the Hilbert method is more direct and leads to a solution which is in a form where the solution properties can be simply deduced. A recent application of the method given in Stoker's book is in Ehrenmark (1992) where a homogeneous difference equation is solved. An alternative would be to evaluate the difference equation by looking at the position and order of its poles and zeros. However, it is more convenient to treat it as a Hilbert problem and use methods based on the Plemelj formulae, i.e. Bantsuri (1973). Without loss of generality we can rewrite (A1) as

$$X(s+\alpha) = K(s)\,X(s), \tag{A2}$$

where $K(s) \to 1$ [and $\log K(s)$ is such that the integral in (A7) converges] as $|s| \to \infty$ in the strip $-\alpha \leqslant \Re(s) < \alpha$. This is done by determining the asymptotic behaviour of $F(s)$ and then letting $A(s) = H(s)X(s)$ with $K(s) = H(s)F(s)/H(s+\alpha)$. The function $H(s)$ being chosen by inspection. Let us define

$$X(s) = Y(s), \quad 0 \leqslant \Re(s) < \alpha. \tag{A3}$$

We take $Y(s)$ to be a sectionally holomorphic function in the strip cut along the imaginary axis and to be a function of period α. It is then simple to deduce that

$$X(s) = Y(s)/K(s), \quad -\alpha \leqslant \Re(s) < 0. \tag{A4}$$

Now $X(s)$ is continuous across the cut, so we define the functions with subscript plus (and minus) to denote the limiting value of the function as it approaches the cut from the left (right). We deduce the following Plemelj formulae for the periodic function $Y(s)$

$$\log Y_+(s) + \log Y_-(s) = 2\log Y_p(s) \tag{A5}$$

$$\log Y_+(s) - \log Y_-(s) = \log K(s) \tag{A6}$$

hence

$$\log Y_p(s) = \frac{1}{2i\alpha}\int_{-i\infty}^{i\infty} \cot\frac{\pi(t-s)}{\alpha}\log K(t)\,\mathrm{d}t \tag{A7}$$

with this integral taken to be Cauchy principal valued. The integral (A7) is the usual Cauchy integral formula mapped into a strip with periodicity conditions assumed. The solution is unique for $\log Y_p(s)$ under the assumption that $Y(s)$ tends to 1 as $|s| \to \infty$. We note that with the above definition of $K(s)$, $\log K(s)$ is continuous on the imaginary axis and we assume it satisfies a Hölder condition on the line and at infinity. Having found the solution to the first-order difference equation in the strip $-\alpha \leqslant \Re(s) < \alpha$ it is a simple manner to shift this strip to the left or right using (A1) thereby recovering the solution for any s.

A similar method can be used for the particular solutions of the inhomogeneous difference equations; for the difference equation

$$A_0(s+\alpha) - A_0(s) = f(s) \tag{A8}$$

we have the result that

$$A_0(s) = \frac{1}{2i\alpha}\int_{-i\infty}^{i\infty} f(t)\cot\frac{\pi(t-s)}{\alpha}\,\mathrm{d}t, \quad 0 \leqslant \Re(s) < \alpha \tag{A9}$$

$$A_0(s) = \frac{1}{2i\alpha}\int_{-i\infty}^{i\infty} f(t)\cot\frac{\pi(t-s)}{\alpha}\,\mathrm{d}t - f(s), \quad -\alpha \leqslant \Re(s) < 0. \tag{A10}$$

The above integrals are Cauchy principal valued. If $f(s)$ has exponential growth at infinity these integrals diverge. However, as we require $A_0(-\frac{1}{2})-A_0(0)$, say, the resulting integrals converge.

The above formulae are related to the product and sum split formulae used during the application of the Wiener–Hopf technique (Nuller, 1976). If we take the limit $\alpha \to \infty$ the two strips, $-\alpha \leqslant \Re(s) < 0$, $0 \leqslant \Re(s) < \alpha$ tend to two half planes and the periodic function $Y(s)$ becomes sectionally holomorphic in each half plane. The formulae for $X(s)$ and for $A_0(s)$ are then the familiar equations for splitting a function into either a product or sum of functions analytic in each half plane. The imaginary axis becomes a line of common analyticity for the functions defined in each half plane and this allows the Wiener–Hopf procedure to be followed. The limit as $\alpha \to -\infty$ can be considered in a similar manner.